Algebraic entropies of commuting endomorphisms of torsion abelian groups

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Respectfully dedicated to the 95-th birthday of Lászlo Fuchs

Abstract

For actions of m commuting endomorphisms of a torsion abelian group we compute the algebraic entropy and the algebraic receptive entropy, showing that the latter one takes finite positive values in many cases when the former one vanishes.

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1 Introduction

A *left semigroup action* $S \overset{\alpha}{\curvearrowright} A$ of a semigroup S on an abelian group A (*by group endomorphisms*) is defined by $\alpha : S \times A \to A$, $(s,x) \mapsto \alpha(s)(x)$ with $\alpha(st) = \alpha(s) \circ \alpha(t)$ and $\alpha(s)(x+y) = \alpha(s)(x) + \alpha(s)(y)$ for every $s,t \in S$ and every $x,y \in A$. In case S is a monoid with neutral element e, we impose also $\alpha(e)(x) = x$ for every every $x \in A$. For $N \subseteq S$ and $F \subseteq A$, let $T_N(\alpha,F) = \sum_{s\in N} \alpha(s)(F)$. Let $\mathcal{F}(A)$ denote the family of all finite subgroups of A.

We recall that a *right Følner sequence* of a semigroup S is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite non-empty subsets of S such that $\lim_{n\to\infty} |F_n s \setminus F_n|/|F_n| = 0$ for every $s \in S$. A countable semigroup S is *right-amenable* if and only if S admits a right Følner sequence. Consider an action $S \overset{\alpha}{\curvearrowright} A$ of a countable cancellative right-amenable semigroup S on an abelian group A. For $F \in \mathcal{F}(A)$ let

$$H_{alg}(\alpha,F) = \lim_{n\to\infty} \frac{\log |T_{N_n}(\alpha,F)|}{|N_n|},$$

where $(N_n)_{n\in\mathbb{N}}$ is any right Følner sequence of S (the limit exists, it is finite and does not depend on the right Følner sequence). The *algebraic entropy* of α is $\mathrm{ent}(\alpha) = \sup\{H_{alg}(\alpha,F) \colon F \in \mathcal{F}(A)\}$ [3]. This concept extends in a natural way the algebraic entropy ent for $\mathbb{N}$-actions on A, that is, for group endomorphisms of A, introduced in [1, 10] and developed in [5].

The nice properties of the algebraic entropy ent stem from the fact that the action $S \overset{\alpha}{\curvearrowright} A$ provides a left $\mathbb{Z}[S]$-module structure on A: ent is an invariant of the category $\mathfrak{T}_S$ of left $\mathbb{Z}[S]$-modules that are torsion as abelian groups, and it is furthermore a length function of $\mathfrak{T}_S$ in the sense of Northcott and Reufel, and of Vámos (see Fact 2.5). Moreover, there is a remarkable connection of ent with the topological entropy, that is, $\mathrm{ent}(\alpha)$ coincides with the topological entropy of the dual action of α (i.e., the action induced by α of S on the Pontryagin dual group $\widehat{A}$ of A) [3, 10].

On the other hand, ent presents some shortcomings from intuitive point of view. For example, if S is finite then $\mathrm{ent}(\alpha) = \log |A|/|S|$, so in particular $\mathrm{ent}(\alpha) = \infty$ whenever A is infinite. Moreover, there are many cases when S is a group, T is a subgroup of S and the restricted action $\alpha\restriction_T$ has $\mathrm{ent}(\alpha\restriction_T) = \infty$ while $\mathrm{ent}(\alpha) = 0$; in case T is a normal subgroup of S always $\mathrm{ent}(\alpha) \leq \mathrm{ent}(\alpha\restriction_T)$. If again S is a group and T is a normal subgroup of S with $T \subseteq \ker\alpha$, then $\mathrm{ent}(\alpha) \leq \mathrm{ent}(\bar\alpha_{S/T})$, where $\bar\alpha_{S/T}$ is the quotient action induced by α; also in this case the inequality can be strict.

This suggested us to consider the new option for algebraic entropy offered in Definition 1.2 (see [2]), and inspired by [7], where counterparts of the bizzarre behavior of the topological entropy were pointed out. Following [7], a *regular system* of a finitely generated monoid S is a sequence

$\Gamma = (N_n)_{n\in\mathbb{N}}$ of finite subsets of S such that $N_0 = \{e\}$ and $N_i N_j \subseteq N_{i+j}$ for every $i, j \in \mathbb{N}$. In particular, $N_n \subseteq N_{n+1}$ for every $n \in \mathbb{N}$. The regular system $\Gamma = (N_n)_{n\in\mathbb{N}}$ is *exhaustive* if $S = \bigcup_{n\in\mathbb{N}} N_n$ and it is *standard* if for every $n \in \mathbb{N}$ the set $N_n = N_1 \dots N_1$ is the n-fold setwise product of N_1, and we set $N_0 = \{e\}$, by definition.

Example 1.1. The following is the most natural example of an exhaustive standard regular system. If the monoid S is finitely generated by N_1, then the standard regular system $(N_n)_{n\in\mathbb{N}}$ of S is exhaustive (e.g., $S = \mathbb{N}$ with $N_n = \{0, \dots, n\}$ for every $n \in \mathbb{N}$).

Definition 1.2. Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, A a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. For $F \in \mathcal{F}(A)$, let

$$\widetilde{H}^{\Gamma}_{alg}(\alpha, F) = \limsup_{n\to\infty} \frac{\log |T_{N_n}(\alpha, F)|}{n}.$$

The *algebraic receptive entropy of* α *with respect to* Γ is

$$\widetilde{\mathrm{ent}}^{\Gamma}(\alpha) = \sup\{\widetilde{H}^{\Gamma}_{alg}(\alpha, F) \colon F \in \mathcal{F}(A)\}.$$

Unlike the case of ent, it may occur that $\widetilde{H}^{\Gamma}_{alg}(\alpha, F) = \infty$ for some $F \in \mathcal{F}(A)$ (see the proof of Theorem 5.1).

Also the algebraic receptive entropy extends in a natural way the algebraic entropy ent for $\mathbb{N}$-actions, as we show in §2, where we recall the basic properties of the algebraic (receptive) entropy that we use further on.

In §3 we make use of the standard correspondence between $\mathbb{N}^m$-actions on abelian groups A of prime exponent p and R_p-module structures on A, where $R_p = \mathbb{F}_p[X_1, \dots, X_m]$ is the ring of polynomials of m variables $X_1, \dots, X_m$ over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. This allows us to freely pass from $\mathbb{N}^m$-actions to R_p-modules and viceversa, writing $\mathrm{ent}(A)$ or $\widetilde{\mathrm{ent}}(A)$ for an R_p-module A, having in mind the algebraic (receptive) entropy of the corresponding $\mathbb{N}^m$-action. As a first step we compute the algebraic (receptive) entropy of R_p (see Theorem 3.3).

In §4 we compute the algebraic entropy ent. First we see that $\mathrm{ent}(A) = 0$ when $A = R_p/\mathfrak{a}$ is an infinite cyclic R_p-module and $\mathfrak{a}$ is a non-zero ideal of R_p (see Theorem 4.1). Using this result we prove that $\mathrm{ent}(A) = \mathrm{rank}_{R_p}(A) \log p$ for an arbitrary R_p-module A (see Theorem 4.4). This allows us to obtain a characterization of $\mathbb{N}^m$-actions on torsion abelian groups with zero algebraic entropy via Bernoulli shifts (see Proposition 4.7).

The first step in §5 is that the receptive algebraic entropy of an infinite cyclic R_p-module $A = R_p/\mathfrak{a}$ is infinite whenever $\mathfrak{a} \neq 0$ is a principal ideal of R_p (see Theorem 5.1). Moreover, in the case $m = 2$ we compute $\widetilde{\mathrm{ent}}(A)$ even when $\mathfrak{a}$ is not necessarily principal (see Theorem 5.5 and Corollary 5.7). Remaining in the case $m = 2$, we describe when $0 < \widetilde{\mathrm{ent}}(A) < \infty$ for a finitely generated R_p-module A (see Corollary 5.8).

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2 Basic properties and examples

Let S be a finitely generated monoid and $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S. For $m \in \mathbb{N}_+$, the direct product S^m carries two regular systems extending Γ in a natural way:

(i) the *Cartesian extension* of Γ is $\Gamma^m = (N_n^m)_{n\in\mathbb{N}}$;

(ii) if N_1 is a finite set of generators of S containing e, the *minimal extension* $\Gamma^{(m)}$ of Γ is the standard regular system $\Gamma^{(m)} = (N_n^{(m)})_{n\in\mathbb{N}}$ of S^m with $N_1^{(m)} = \{(s, e, \dots, e) \colon s \in N_1\} \cup \dots \cup \{(e, \dots, e, s) \colon s \in N_1\}$.

In $S=\mathbb{N}$ consider the exhaustive standard regular system $\Xi=(\Xi_n)_{n\in\mathbb{N}}$ with $\Xi_1=\{0,1\}$ (so $\Xi_n=\{0,\dots,n\}$ for every $n\in\mathbb{N}$). Obviously, this is the smallest exhaustive standard regular system of $\mathbb{N}$.

For $\mathbb{N}^m$ the Cartesian extension of Ξ is $\Xi^m=(\Xi^m_n)_{n\in\mathbb{N}}$ with $\Xi^m_1=\{0,1\}^m$, and the minimal extension of Ξ is $\Xi^{(m)}=(\Xi^{(m)}_n)_{n\in\mathbb{N}}$ with $\Xi^{(m)}_n=\{(a_1,\dots,a_m)\in\mathbb{N}^m\colon a_1+\ldots+a_m\leq n\}$ for every $n\in\mathbb{N}$.

Remark 2.1. Let S be a finitely generated monoid, A an abelian group and $S\overset{\alpha}{\curvearrowright}A$. If H is a submonoid of S and $\Gamma=(N_n)_{n\in\mathbb{N}}$ is a regular system of H, then $\Gamma=(N_n)_{n\in\mathbb{N}}$ is a regular system of S as well, and $\widetilde{H}^\Gamma_{alg}(\alpha\restriction_H,F)=\widetilde{H}^\Gamma_{alg}(\alpha,F)$ for every $F\in\mathcal{F}(A)$, so also $\widetilde{\mathrm{ent}}^\Gamma(\alpha\restriction_H)=\widetilde{\mathrm{ent}}^\Gamma(\alpha)$.

The above notions of algebraic entropy and algebraic receptive entropy extend the usual notion of algebraic entropy of a group endomorphism:

Remark 2.2. (a) Let A be an abelian group and $\phi:A\to A$ an endomorphism. We recall from [4, 5, 10] that, for $F\in\mathcal{F}(A)$, $T_0(\phi,F)=\{0\}$ and $T_n(\phi,F)=F+\phi(F)+\ldots+\phi^{n-1}(F)$ for every $n\in\mathbb{N}_+$; moreover

$$H_{alg}(\phi,F)=\lim_{n\to\infty}\frac{\log|T_n(\phi,F)|}{n},$$

and the limit exist by Fekete Lemma. The algebraic entropy from [10] is $\mathrm{ent}(\phi)=\sup\{H_{alg}(\phi,F)\colon F\in\mathcal{F}(A)\}$. Consider the action $\mathbb{N}\overset{\alpha_\phi}{\curvearrowright}A$ defined by $\alpha_\phi(1)=\phi$. It is easy to see that $T_{\Xi_n}(\alpha_\phi,F)=T_{n+1}(\phi,F)$ for every $n\in\mathbb{N}$ and $F\in\mathcal{F}(A)$, and so $\mathrm{ent}(\phi)=\widetilde{\mathrm{ent}}^\Xi(\alpha_\phi)=\mathrm{ent}(\alpha_\phi)$.

(b) If $\phi:A\to A$ is an isomorphism, ϕ induces also an action $\mathbb{Z}\overset{\beta_\phi}{\curvearrowright}A$ defined by $\beta_\phi(1)=\phi$. Using again the standard regular system Ξ of $\mathbb{Z}$, by Remark 2.1 and item (a) we find $\mathrm{ent}(\phi)=\widetilde{\mathrm{ent}}^\Xi(\beta_\phi)$.

Since Ξ is not exhaustive for $\mathbb{Z}$, it makes sense to consider the exhaustive standard regular system $\Xi'=(\Xi'_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}$ with $\Xi'_n=\{-n,\dots,n\}$ for every $n\in\mathbb{N}$. Then, for every $n\in\mathbb{N}$ and $F\in\mathcal{F}(A)$,

$$T_{\Xi'_n}(\beta_\phi,F)=\phi^{-n}(F)+\ldots+F+\phi(F)+\ldots+\phi^n(F)=T_{n+1}(\phi^{-1},F)+\phi(T_n(\phi,F)).$$

Since $\phi^n(T_{\Xi'_n}(\beta_\phi,F))=T_{2n+1}(\phi,F)$, and ϕ is bijective, we have $|T_{\Xi'_n}(\beta_\phi,F)|=|T_{2n+1}(\phi,F)|$. Therefore,

$$\begin{aligned}\widetilde{H}^{\Xi'}_{alg}(\beta_\phi,F)&=\limsup_{n\to\infty}\frac{\log|T_{\Xi'_n}(\beta_\phi,F)|}{n}=\limsup_{n\to\infty}\frac{\log|T_{2n+1}(\phi,F)|}{n}=\\&=\lim_{n\to\infty}\frac{2n+1}{n}\frac{\log|T_{2n+1}(\phi,F)|}{2n+1}=2H_{alg}(\phi,F).\end{aligned}$$

We conclude that $\widetilde{\mathrm{ent}}^{\Gamma'}(\beta_\phi)=2\mathrm{ent}(\phi)$.

Example 2.3. Every exhaustive standard regular system $\Gamma=(N_n)_{n\in\mathbb{N}}$ of $\mathbb{N}^m$ gives rise to a right Følner sequence of $\mathbb{N}^m$. The same occurs for every finitely generated monoid of subexponential growth.

(We have no example of a regular system of an amenable finitely generated group that fails to be a right Følner sequence.)

Remark 2.4. Let S be a finitely generated monoid, A an abelian group and $S\overset{\alpha}{\curvearrowright}A$. Suppose that $\Gamma=(N_n)_{n\in\mathbb{N}}$ is a regular system of S which is also a right Følner sequence of S and such that

$$|N_n|\geq cn^2 \text{ for some constant } c>0 \text{ and all } n\in\mathbb{N}_+. \tag{2.1}$$

It follows from the definitions that, if for some $F\in\mathcal{F}(A)$ we have $\widetilde{H}^\Gamma_{alg}(\alpha,F)<\infty$, then $H_{alg}(\alpha,F)=0$. Thus, $\widetilde{\mathrm{ent}}^\Gamma(\alpha)<\infty\ \Rightarrow\ \mathrm{ent}(\alpha)=0$ (equivalently, $\mathrm{ent}(\alpha)>0\ \Rightarrow\ \widetilde{\mathrm{ent}}^\Gamma(\alpha)=\infty$).

The condition (2.1) is available for example whenever S is a finitely generated group that does not contain a cyclic subgroup of finite index.

We recall basic properties of ent that we use below. For a monoid S, abelian groups A,B and actions $S\overset{\alpha}{\curvearrowright}A, S\overset{\beta}{\curvearrowright}B$, α and β are *conjugated* by an isomorphism $\xi:A\to B$ if $\xi\circ\alpha(g)=\beta(g)\circ\xi$ for every $g\in S$.

Fact 2.5. *[3] Let S be a cancellative right-amenable monoid, A a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. Let B be an α-invariant subgroup of A and denote by α_B and $\alpha_{A/B}$ the induced actions of S on B and on A/B, respectively.*

(Invariance) For $S \overset{\beta}{\curvearrowright} C$ with C a torsion abelian group, if α and β are conjugated, then $\mathrm{ent}(\alpha) = \mathrm{ent}(\beta)$.

(Monotonicity) $\mathrm{ent} \geq \max\{\mathrm{ent}(\alpha_B), \mathrm{ent}(\alpha_{A/B})\}$.

(Continuity) If A is a direct limit of α-invariant subgroups $\{A_i : i \in I\}$, then $\mathrm{ent}(\alpha) = \sup_{i\in I} \mathrm{ent}(\alpha_{A_i})$.

(weak Addition Theorem) If $A = A_1 \times A_2$, with A_1, A_2 α-invariant subgroups of A, then $\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_{A_1}) + \mathrm{ent}(\alpha_{A_2})$.

(Addition Theorem) $\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_B) + \mathrm{ent}(\alpha_{A/B})$.

The next theorem was inspired by a similar result for $\mathbb{N}$-actions in [5].

Theorem 2.6. *Let S be a countable cancellative right-amenable monoid, A a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. Then $\mathrm{ent}(\alpha) > 0$ if and only if there exists a prime p such that $\mathrm{ent}(\alpha_{A[p]}) > 0$.*

Proof. In view of the monotonicity of ent, it is enough to prove the necessity.

We consider first the case when A is a bounded p-group for some prime p; let p^k be the exponent of A. If $k = 1$ there is nothing to prove. Assume that $k > 1$ and the assertions is true for $k-1$. If $\mathrm{ent}(\alpha_{A[p]}) > 0$, there is nothing to prove. Assume that $\mathrm{ent}(\alpha_{A[p]}) = 0$. Consider the short exact sequence $0 \to A[p] \to A \to pA \to 0$. According to the Addition Theorem and the invariance, $\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_{pA}) + \mathrm{ent}(\alpha_{A[p]}) = \mathrm{ent}(\alpha_{pA})$. Since $p^{k-1}pA = 0$, we can conclude that $\mathrm{ent}(\alpha_{(pA)[p]}) > 0$. Since $(pA)[p]$ is a subgroup of $A[p]$, monotonicity $\mathrm{ent}(\alpha_{A[p]}) > 0$, a contradiction.

If A is a p-group for some prime p, $A = \bigcup_{n\in\mathbb{N}} A[p^n]$ and $\mathrm{ent}(\alpha) = \sup_{n\in\mathbb{N}} \mathrm{ent}(\alpha_{A[p^n]})$. Hence, our hypothesis $\mathrm{ent}(\alpha) > 0$ yields $\mathrm{ent}(\alpha_{A[p^n]}) > 0$ for some $n \in \mathbb{N}$. The above argument applied to $A[p^n]$, combined with the obvious equality $(A[p^n])[p] = A[p]$, entails $\mathrm{ent}(\alpha_{A[p]}) > 0$.

Let $A = \bigoplus_p A_p$, where each A_p is a p-group. Then using the actions $S \overset{\alpha_{A_p}}{\curvearrowright} A_p$, one has $\mathrm{ent}(\alpha) = \sum_p \mathrm{ent}(\alpha_{A_p})$ by Fact 2.5. Hence, $\mathrm{ent}(\alpha) > 0$ yields the existence of a prime p such that $\mathrm{ent}(\alpha_{A_p}) > 0$. Now the above argument gives $\mathrm{ent}(\alpha_{A_p[p]}) > 0$. Since $A_p[p] = A[p]$, we are done. $\square$

Fact 2.7. *[2] Invariance, monotonicity and continuity remain valid also for the algebraic receptive entropy, while in the weak Addition Theorem only one inequality is available: if $A = A_1 \times A_2$, with A_1, A_2 α-invariant subgroups of A, then* $\widetilde{\mathrm{ent}}^{\Gamma}(\alpha) \leq \widetilde{\mathrm{ent}}^{\Gamma}(\alpha_{A_1}) + \widetilde{\mathrm{ent}}^{\Gamma}(\alpha_{A_2})$.

3 $\mathbb{N}^m$-actions vs $\mathbb{Z}[X_1, \ldots, X_m]$-modules

Notation 3.1. For an action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ on an abelian group A, from now on, we simply use the notation as follows: $\widetilde{H}_{alg} = \widetilde{H}_{alg}^{\Xi^{(m)}}$ and $\widetilde{\mathrm{ent}} = \widetilde{\mathrm{ent}}^{\Xi^{(m)}}$.

Following a well-known approach of Kaplansky, an $\mathbb{N}^m$-action on an abelian group A can be viewed in a standard way as an R_0-module structure on A, where $R_0 = \mathbb{Z}[X_1, \ldots, X_m]$ is the ring of polynomials of m variables $X_1, \ldots, X_m$ over $\mathbb{Z}$. Similarly, a $\mathbb{Z}^m$-action on A can be viewed as a module structure on A over the ring $\mathbb{Z}[X_1^{\pm1}, \ldots, X_m^{\pm1}]$ of Laurent polynomials of m variables $X_1, \ldots, X_m$ over $\mathbb{Z}$.

In case A has a prime exponent p, one can use also the ring $R_p = \mathbb{F}_p[X_1, \ldots, X_m]$ of polynomials of m variables $X_1, \ldots, X_m$ over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and provide as above an obvious connection between $\mathbb{N}^m$-actions and R_p-module structures on A. Moreover, the $\mathbb{Z}^m$-actions on A can be viewed as module structures on A over the ring $\mathbb{F}_p[X_1^{\pm1}, \ldots, X_m^{\pm1}]$ of Laurent polynomials of m variables $X_1, \ldots, X_m$ over $\mathbb{F}_p$. In the sequel we freely pass from $\mathbb{N}^m$-actions to R_p-modules and viceversa, writing $\mathrm{ent}(A)$ or $\widetilde{\mathrm{ent}}(A)$ for an R_p-module A, having in mind the algebraic (receptive) entropy of the corresponding $\mathbb{N}^m$-action.

This approach is efficient in the case $m = 1$, when R_p is a principal ideal domain, so R_p-modules have a relatively simple structure and one can easily prove that $\mathrm{ent}(A) = \widetilde{\mathrm{ent}}(A) =$

$\mathrm{rank}_{R_p}(A)$, where $\mathrm{rank}_{R_p}(A)$ denotes the maximum size of a subset of A independent over R_p. This characterization is extended to the case $m > 1$ in Theorem 4.3. As a starting point, we consider the cyclic R_p-module $A = R_p$ in Theorem 3.3.

Remark 3.2. For the sake of simplicity, it makes sense to replace, whenever necessary, the additive monoid $(\mathbb{N}^m, +, 0)$ by the moltiplicative submonoid $M = \{X_1^{s_1} \dots X_m^{s_m} : (s_1, \dots, s_m) \in \mathbb{N}^m\}$ of the multiplicative monoid $(R_p, \cdot, 1)$ of the ring $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. For $b = X_1^{s_1} \dots X_m^{s_m} \in M$, the *degree* of b is $d(b) = \sum_{i=1}^m s_i$.

Substantially, for an action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ on an abelian group A of exponent a prime p, the commuting endomorphisms $\phi_i := \alpha(e_i)$ of A, where e_i is the i-th member of the canonical base of $\mathbb{N}^m$, make it become an R_p-module, as already explained above. Now the n-th member $\Xi_n^{(m)}$ of the standard minimal regular system $\Xi^{(m)}$ of $\mathbb{N}^m$ obviously corresponds to

$$B_n = \{b \in M : d(b) \leq n\} \subseteq M.$$

Let

$$b_n = |B_n| = |\Xi_n^{(m)}| \tag{3.1}$$

and note that b_n coincides with the so called $n+1$-th simplicial m-polytopic number known to be equal to the binomial coefficient C_m^{n+m}. Hence,

$$b_n = \frac{1}{m!} n^m + \frac{m+1}{2(m-1)!} n^{m-1} + \dots \tag{3.2}$$

is a polynomial of n of degree m, and so $\lim_{n\to\infty} b_n/n = \infty$ whenever $m > 1$.

Theorem 3.3. *If $m \in \mathbb{N}_+$, then $\mathrm{ent}(R_p) = \log p$. If $m = 1$, then $\widetilde{\mathrm{ent}}(R_p) = \mathrm{ent}(R_p) = \log p$, otherwise $\widetilde{\mathrm{ent}}(R_p) = \infty > \mathrm{ent}(R_p)$.*

Proof. If $m = 1$ the equality $\widetilde{\mathrm{ent}}(R_p) = \mathrm{ent}(R_p)$ follows from Remark 2.2(a), while the equality $\mathrm{ent}(R_p) = \log p$ is well known since the corresponding $\mathbb{N}$-action is the Bernoulli shift $(x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, x_2, \dots)$ of $\bigoplus_{\mathbb{N}} \mathbb{Z}_p$ [5]. Assume $m > 1$ and put $F = \mathbb{F}_p \in \mathcal{F}(A)$, that is, F is the set of all p polynomials of degree 0 and the zero polynomial. By (3.1), $|\Xi_n^{(m)}| = |B_n| = b_n$. Since $T_{\Xi_n^{(m)}}(\alpha, F) =: V_n$ is a vector space and B_n is a base of V_n, then $\dim V_n = |B_n| = b_n$, so $|V_n| = p^{b_n}$ and

$$\log |V_n| = b_n \log p. \tag{3.3}$$

Since $(B_n)_{n\in\mathbb{N}}$ is a Følner sequence of R_p, we deduce that

$$H_{alg}(\alpha, F) = \lim_{n\to\infty} \frac{\log |V_n|}{|\Xi_n^{(m)}|} = \lim_{n\to\infty} \frac{b_n \log p}{b_n} = \log p.$$

If we replace F by $V_l = T_{\Xi_l^{(m)}}(\alpha, F)$ for some $l \in \mathbb{N}$, we have $T_{\Xi_n^{(m)}}(\alpha, V_l) = T_{\Xi_{n+l}^{(m)}}(\alpha, F) = V_{n+l}$, which leads to $H_{alg}(\alpha, V_l) = H_{alg}(\alpha, F) = \log p$. Since every finite subgroup of R_p is contained in some V_l, this proves $\mathrm{ent}(R_p) = \log p$.

Since $\lim_{n\to\infty} |N_n|/n = \lim_{n\to\infty} b_n/n = \infty$ by Remark 3.2, $\widetilde{H}_{alg}(\alpha, F) = \infty$, and therefore $\widetilde{\mathrm{ent}}(\alpha) = \infty$. □

4 The algebraic entropy of $\mathbb{F}_p[X_1, \dots, X_m]$-modules

Here we consider $\mathbb{N}^m$-actions on an abelian group A of exponent a prime p; so let $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. Moreover, we use $\Xi^{(m)}$, which is an exhaustive standard regular system of $\mathbb{N}^m$ and also a Følner sequence. Since $\mathrm{ent}(A) = 0$ for finite abelian groups A, we assume in the sequel that A is infinite.

Theorem 4.1. *If $A = R_p/\mathfrak{a}$ is an infinite cyclic R_p-module, where $\mathfrak{a} \neq 0$ is an ideal of R_p, then $\mathrm{ent}(A) = 0$.*

Proof. Denote by $\overline{\alpha}$ the $\mathbb{N}^m$-action corresponding to the R_p-module structure of $A = R_p/\mathfrak{a}$, and let $q : R_p \to A = R_p/\mathfrak{a}$ be the quotient map. We keep the notation from the proof of Theorem 3.3. In particular, $V_n = T_{\Xi_n^{(m)}}(\alpha, F)$ is a linear subspace of R_p for $n \in \mathbb{N}$, so $q(V_n) = T_{\Xi_n^{(m)}}(\overline{\alpha}, q(\mathbb{F}_p))$

is a linear subspace of A. Since $q(V_n) \cong V_n/V_n \cap \ker q = V_n/V_n \cap \mathfrak{a}$, so $|q(V_n)| = |V_n/V_n \cap \mathfrak{a}| = |V_n|/|V_n \cap \mathfrak{a}|$, and hence

$$\log|q(V_n)| = \log|V_n| - \log|V_n \cap \mathfrak{a}|. \tag{4.1}$$

Consider the principal ideal $\mathfrak{a} = (a(X_1,\dots,X_m))$ and let d be the degree of a. Then $p(X_1,\dots,X_m) \in V_n \cap \mathfrak{a}$ precisely when $p(X_1,\dots,X_m) = r(X_1,\dots,X_m)a(X_1,\dots,X_m)$ for some $r(X_1,\dots,X_m) \in R_p$ of degree at most $n-d$, i.e., $r(X_1,\dots,X_m) \in V_{n-d}$. Since R_p is a domain, the map

$$V_{n-d} \ni r(X_1,\dots,X_m) \mapsto r(X_1,\dots,X_m)a(X_1,\dots,X_m) \in V_n \cap \mathfrak{a}$$

provides a bijection, so $|V_n \cap \mathfrak{a}| = |V_{n-d}|$. Hence, (3.2), (3.3) and (4.1) give

$$\log|q(V_n)| = (b_n - b_{n-d})\log p = \left(\frac{d}{(m-1)!}n^{m-1} + \dots\right)\log p; \tag{4.2}$$

in particular $\log|q(V_n)|$ is a polynomial of n of degree $m-1$. This implies

$$H_{alg}(\overline{\alpha}, q(\mathbb{F}_p)) = 0. \tag{4.3}$$

For any $F' \in \mathcal{F}(R_p)$ one can find $n_0 \in \mathbb{N}$ such that $F' \subseteq T_{\Xi^{(m)}_{n_0}}(\alpha, \mathbb{F}_p)$, so

$$T_{\Xi^{(m)}_{n+n_0}}(\alpha, \mathbb{F}_p) \supseteq T_{\Xi^{(m)}_{n}}(\alpha, F') \quad \text{for every} \quad n \in \mathbb{N}. \tag{4.4}$$

For $F^* \in \mathcal{F}(A)$ there is $F' \in \mathcal{F}(R_p)$ with $q(F') = F^*$. By (4.4), this gives

$$T_{\Xi^{(m)}_{n+n_0}}(\overline{\alpha}, q(\mathbb{F}_p)) \supseteq T_{\Xi^{(m)}_{n}}(\overline{\alpha}, F^*). \tag{4.5}$$

Dividing by $|\Xi^{(m)}_n|$ and using (4.3) we deduce that $H_{alg}(\overline{\alpha}, F^*) = 0$ as well. Therefore $\mathrm{ent}(A) = 0$.

In case when $\mathfrak{a}$ is not necessarily principal, find a principal ideal $0 \neq \mathfrak{b} \subseteq \mathfrak{a}$. Then $\mathrm{ent}(R_p/\mathfrak{b}) = 0$ by the above argument. Since $A = R_p/\mathfrak{a}$ is a quotient of $R_p/\mathfrak{b}$ by Fact 2.5 we deduce that $\mathrm{ent}(A) = 0$ as well. $\square$

The computation of ent for non-cyclic R_p-modules can be somehow reduced to the case of cyclic ones.

Definition 4.2. Let R be a domain and A be an R-module. Call $a \in A$ *R-torsion* if $\mathrm{ann}(a) = \{r \in R : ra = 0\} \neq 0$. Let $t_R(A)$ denote the R-submodule of A consisting of all R-torsion elements of A. Call A *R-torsion free* (resp., *R-torsion*), if $t_R(A) = 0$ (resp., $t_R(A) = A$).

Clearly, $A/t_R(A)$ is R-torsion free. The next result shows that we can study ent in R_p-torsion free modules A.

Lemma 4.3. *Let A be an R_p-module. Then* $\mathrm{ent}(t_{R_p}(A)) = 0$ *and* $\mathrm{ent}(A) = \mathrm{ent}(A/t_{R_p}(A))$*. In particular* $\mathrm{ent}(A) = 0$ *if A is R_p-torsion.*

Proof. If $a \in A$ is R_p-torsion, then $\mathrm{ann}(a) \neq 0$ and $aR_p \cong R_p/\mathrm{ann}(a)$. Hence $\mathrm{ent}(aR_p) = 0$ either because it is finite or by Theorem 4.1. By Fact 2.5, $\mathrm{ent}(t_{R_p}(A)) = 0$ and so $\mathrm{ent}(A) = \mathrm{ent}(A/t_{R_p}(A))$. $\square$

Theorem 4.4. *For every R_p-module A,* $\mathrm{ent}(A) = \mathrm{rank}_{R_p}(A)\log p$.

Proof. Every R_p-independent subset of A is contained in some maximal R_p-independent subset X' of A. The submodule A_0 of A generated by X' is free and $\mathrm{rank}_{R_p}(A) = \mathrm{rank}_{R_p}(A_0) = |X'|$. Then A/A_0 is R_p-torsion, so $\mathrm{ent}(A/A_0) = 0$ by Lemma 4.3, and hence $\mathrm{ent}(A) = \mathrm{ent}(A_0)$ by Fact 2.5.

If $\mathrm{rank}_{R_p}(A)$ is infinite, then A_0 contains a submodule $M \cong \bigoplus_{\mathbb{N}} R_p$, and so $\mathrm{ent}(A_0) \geq \mathrm{ent}(M) = \infty$ by Fact 2.5. If $\mathrm{rank}_{R_p}(A) = t$ is finite, then $A_0 = R_p^t$, and so $\mathrm{ent}(A_0) = t\,\mathrm{ent}(R_p) = t\log p$ by Theorem 3.3. $\square$

Combining Theorem 4.4 with Theorem 4.1 one obtains the following.

Corollary 4.5. *For an infinite R_p-module A,* $\mathrm{ent}(A) = 0$ *if and only if* $\mathrm{rank}_{R_p}(A) = 0$.

We aim to obtain a counterpart of Corollary 4.5 for an arbitrary torsion abelian group A, but the condition $\mathrm{rank}_{R_p}(A) = 0$ becomes meaningless, so we give an alternative characterization of the property $\mathrm{ent}(A) = 0$.

Remark 4.6. Let A be an R_p-module, corresponding to the action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$. According to Corollary 4.5, $\mathrm{rank}_{R_p}(A) > 0$ if and only if A contains a submodule B isomorphic to $R_p \cong \bigoplus_{\mathbb{N}^m} \mathbb{Z}_p$. Since the action of $\mathbb{N}^m \overset{\alpha_B}{\curvearrowright} B$ coincides with the m-dimensional Bernoulli shift over $\mathbb{Z}_p$ (i.e., the m-th Cartesian power of the usual one-dimensional Bernoulli shift – see the proof of Theorem 3.3), we shall refer to this circumstance by simply saying that $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ "contains an m-dimensional Bernoulli shift over" $\mathbb{Z}_p$. In these terms, Corollary 4.5 says that *an action* $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ *on an abelian group* A *of exponent a prime* p *has* $\mathrm{ent}(A) = 0$ *if and only if* $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ *does not contain any* m*-dimensional Bernoulli shift over* $\mathbb{Z}_p$.

Obviously, this terminolgy can be used also when the torsion abelian group A is not necessarily an R_p-module; in such a case, for a prime p, by saying that $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ contains an m-dimensional Bernoulli shift over $\mathbb{Z}_p$ we mean that A contains an α-invariant subgroup $B \cong \bigoplus_{\mathbb{N}^m} \mathbb{Z}_p$ such that $\mathbb{N}^m \overset{\alpha_B}{\curvearrowright} B$ is conjugated to the m-dimensional Bernoulli shift over $\mathbb{Z}_p$.

By means of Theorem 2.6, we obtain the following extension.

Proposition 4.7. *Let* $m \in \mathbb{N}_+$, *let* A *be a torsion abelian group and* $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$. *Then* $\mathrm{ent}(\alpha) > 0$ *if and only if there exists a prime* p *such that* $\mathbb{N}^m \overset{\alpha_{A[p]}}{\curvearrowright} A$ *contains an* m*-dimensional Bernoulli shift over* $\mathbb{Z}_p$.

Proof. According to Theorem 2.6, $\mathrm{ent}(\alpha) > 0$ if and only if $\mathrm{ent}(\alpha_{A[p]}) > 0$ for a prime p. Now apply Corollary 4.5 and Remark 4.6 to $\alpha_{A[p]}$. $\square$

The algebraic entropy of m commuting endomorphisms $\phi_1, \dots, \phi_m$ of an abelian p-group A was already studied in [5]. Since A is a module over the ring $\mathbb{J}_p$ of p-adic integers, one obtains also a natural structure of a $\mathbb{J}_p[X_1, \dots, X_m]$-module on A. If $\mathrm{ent}(\phi_1) = \ldots = \mathrm{ent}(\phi_m) = 0$, then $\mathrm{ent}(\psi) = 0$ for every $\psi \in \mathbb{J}_p[\phi_1, \dots, \phi_m]$ by [5, Lemma 2.5]. Let us see that $\mathrm{ent}(A) = 0$ as well. Indeed, if $\mathrm{ent}(A) > 0$, then Proposition 4.7 provides an m-dimensional Bernoulli shift over $\mathbb{Z}_p$ in A, i.e., a submodule $B \cong \mathbb{F}_p[X_1, \dots, X_m]$. Since B is ϕ_1-invariant and $\phi_1 \restriction_B$ is conjugated to the multiplication by X_1, by Fact 2.5 $\mathrm{ent}(\phi_1) \geq \mathrm{ent}(\phi_1 \restriction_B) \geq \log p > 0$, a contradiction.

With a more careful housekeeping, the above argument proves that, if $m > 1$, $\mathrm{ent}(\phi_1) = \infty$ under the assumption that $\mathrm{ent}(A) > 0$ and even more. Taking for simplicity $A = B = \mathbb{J}_p[\phi_1, \dots, \phi_m]$, then $\mathrm{ent}(\psi) = \infty$ for every endomorphism of A induced by the multiplication by any polynomial $\psi \in \mathbb{J}_p[X_1, \dots, X_m]$ of positive degree.

5 The algebraic receptive entropy of $\mathbb{F}_p[X_1, \dots, X_m]$-modules

We start with the computation of $\widetilde{\mathrm{ent}}$ for cyclic R_p-modules, where $R_p = \mathbb{F}_p[X_1, \dots, X_m]$ and $m > 2$.

Theorem 5.1. *If* $m > 2$ *and* $A = R_p/\mathfrak{a}$ *is an infinite cyclic* R_p*-module, where* $\mathfrak{a} \neq 0$ *is a principal ideal of* R_p, *then* $\widetilde{\mathrm{ent}}(A) = \infty$.

Proof. We keep the notation from the proofs of Theorems 3.3 and 4.1. In particular, α is the $\mathbb{N}^m$-action on R_p and $\overline{\alpha}$ is the $\mathbb{N}^m$-action on A determined by the R_p-module structure of A, $\mathfrak{a} = (a)$ with $d(a) = d$; for $n \in \mathbb{N}$, let $V_n = T_{\Xi_n^{(m)}}(\alpha, F)$ and so $q(V_n) = T_{\Xi_n^{(m)}}(\overline{\alpha}, q(\mathbb{F}_p))$, where $q : R_p \to A = R_p/\mathfrak{a}$ is the quotient map. By (4.2), since $m - 1 > 1$ by hypothesis, we conclude that $\widetilde{H}_{alg}(\overline{\alpha}, q(\mathbb{F}_p)) = \infty$, and so $\widetilde{\mathrm{ent}}(\overline{\alpha}) = \infty$. $\square$

The next example shows that the conclusion of the above theorem need not be true if $\mathfrak{a}$ is not principal.

Example 5.2. (a) Let $m = 3$, that is, $R_p = \mathbb{F}_p[X, Y, Z]$, and let $A = R_p/\mathfrak{a} \cong \mathbb{F}_p[X]$, with $\mathfrak{a} = (Y, Z) = (Y) + (Z)$. So $A \cong \mathbb{F}_p[X]$. Denote by α the $\mathbb{N}^3$-action on A induced by the R_p-module structure. This R_p-module induces an A-module structure on A, and we denote by $\bar{\alpha}$ the associated $\mathbb{N}$-action on A. Since A is a quotient of R_p, $\bar{\alpha}$ is a quotient action of α. Therefore,

$\widetilde{\mathrm{ent}}^{\Xi^{(3)}}(\alpha) = \widetilde{\mathrm{ent}}^{\Xi}(\bar{\alpha})$, and $\widetilde{\mathrm{ent}}^{\Xi}(\bar{\alpha}) = \log p < \infty$ since X acts on $A \cong \mathbb{F}_p[X]$ as the Bernoulli shift (see [5]).

(b) Let $m > 2$ and $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. Fix a positive $d < m$ and put $\mathfrak{a}_d = (X_{d+1}, \dots, X_m) = (X_{d+1}) + \dots + (X_m)$, and $A_d = R_p/\mathfrak{a}_d \cong \mathbb{F}_p[X_1, \dots, X_d]$. Denote by α the $\mathbb{N}^m$-action on A_d induced by the R_p-module structure. This R_p-module induces an A_d-module structure on A_d, and we denote by $\bar{\alpha}$ the associated $\mathbb{N}^d$-action on A_d. Since A_d is a quotient of R_p, $\bar{\alpha}$ is a quotient action of α. Therefore $\widetilde{\mathrm{ent}}(\alpha) = \widetilde{\mathrm{ent}}(\bar{\alpha})$ (see [3]), and $\widetilde{\mathrm{ent}}(\bar{\alpha}) = \log p < \infty$ for $d = 1$ while $\widetilde{\mathrm{ent}}(\bar{\alpha}) = \infty$ for $d > 1$ by Theorem 3.3.

We recall some well-known facts regarding $\mathbb{F}_p[X_1, X_2]$ necessary for the proof of the sharper Theorem 5.5.

Fact 5.3. *If R is a principal ideal domain, then an ideal $\mathfrak{a} \neq 0$ of $R[X]$ is prime if and only if one of the following two cases occur:*

(a) $\mathfrak{a} = \langle f(X) \rangle$ for some irreducible element $f(X) \in R[X]$ (two cases are possible here: either $\deg f > 0$ or $f(X) = p$ for some prime $p \in R$);

(b) $\mathfrak{a} = \langle p, f(X) \rangle$ for some some prime $p \in R$ and $f(X) \in R[X]$ such that $\deg f > 0$ and its projection $\bar{f}(X) \in R/pR[X]$ is irreducible; in this case $\mathfrak{a}$ is a maximal ideal of $R[X]$.

For $R = k[Y]$ with k a finite filed, the maximal ideals of $R[X]$ have finite index.

The next theorem is focused on $\mathbb{N}^2$-actions, so now $R_p = \mathbb{F}_p[X_1, X_2]$. We recall that we always consider the regular system $\Xi^{(2)}$, so we omit to write it every time. Our aim is to compute the algebraic receptive entropy of finitely generated $\mathbb{F}_p[X_1, X_2]$-modules. To this end we start with cyclic R_p-modules, recalling that $\widetilde{\mathrm{ent}}(R_p) = \infty$ according to Theorem 3.3.

Lemma 5.4. *Let $\mathfrak{a}$ be a non-trivial ideal of R_p such that $R_p/\mathfrak{a}$ is infinite and cyclic. Then there exists a principal prime ideal $\mathfrak{p}$ of R_p containing $\mathfrak{a}$.*

Proof. We can apply Lasker-Noether Theorem to deduce that

$$\mathfrak{a} = \bigcap_{i=1}^{s} \mathfrak{q}_i, \tag{5.1}$$

where $\mathfrak{q}_i$ are primary ideals of R_p for $i \in \{1, \dots, s\}$. Let $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$, then $\mathfrak{p}_i$ is prime and clearly $\mathfrak{a} \subseteq \mathfrak{q}_i \subseteq \mathfrak{p}_i$. Since R_p is Noetherian, $\mathfrak{q}_i$ is finitely generated, so there exists $k_i \in \mathbb{N}_+$ such that $\mathfrak{p}_i^{k_i} \subseteq \mathfrak{q}_i$.

Suppose that all $\mathfrak{p}_i$ are maximal. Then, using Fact 5.3, we deduce that all $R_p/\mathfrak{p}_i$ are finite. Since $\mathfrak{p}_i$ is finitely generated, $\mathfrak{p}_i/\mathfrak{p}_i^2$ is finitely generated as an $R_p/\mathfrak{p}_i$-module, hence finite. So, $R_p/\mathfrak{p}_i^2$ is finite as well. Arguing by induction, one can see that $R_p/\mathfrak{p}_i^{k_i}$ is finite. Therefore, $R_p/\mathfrak{q}_i$ is finite for all $i \in \{1, \dots, s\}$. From (5.1) we deduce that $R_p/\mathfrak{a}$ embeds into the direct product $\prod_{i=1}^{s} R_p/\mathfrak{q}_i$, that is finite. Hence $A = R_p/\mathfrak{a}$ is finite as well, a contradiction. Then at least one $\mathfrak{p}_i$ is not maximal. Being $\mathfrak{p}_i$ prime and non-maximal, according to Fact 5.3 it is principal. $\square$

Theorem 5.5. *If $A = R_p/\mathfrak{a}$ is an infinite cyclic R_p-module, with $\mathfrak{a} \neq 0$ ideal of R_p, then $0 < \widetilde{\mathrm{ent}}(A) < \infty$. Moreover, when $\mathfrak{a} = (a(X_1, X_2))$ is a principal ideal, then $\widetilde{\mathrm{ent}}(A) = \deg a \cdot \log p$.*

Proof. First assume that $\mathfrak{a} = (a)$ is principal and let $d = d(a)$. Denote by α the $\mathbb{N}^2$-action on R_p and $\overline{\alpha}$ the $\mathbb{N}^2$-action corresponding to the R_p-module structure of $A = R_p/\mathfrak{a}$, moreover let $q : R_p \to A = R_p/\mathfrak{a}$ be the quotient map. Let $F \in \mathcal{F}(A)$ and $F' \in \mathcal{F}(R_p)$ with $q(F') = F$. There exists $n_0 \in \mathbb{N}$ such that $F' \subseteq T_{\Xi^{(2)}_{n_0}}(\alpha, \mathbb{F}_p)$, so as in (4.5), $T_{\Xi^{(2)}_{n+n_0}}(\overline{\alpha}, q(\mathbb{F}_p)) \supseteq T_{\Xi^{(2)}_n}(\overline{\alpha}, F)$ for every $n \in \mathbb{N}$. Since, in the notation of Theorem 4.1, for $V_n = T_{\Xi^{(2)}_n}(\alpha, \mathbb{F}_p)$ one has $q(V_n) = T_{\Xi^{(2)}_n}(\overline{\alpha}, q(\mathbb{F}_p))$, using (4.2) and (4.5), we deduce that for every $n \in \mathbb{N}$,

$$\log |T_{\Xi^{(2)}_n}(\overline{\alpha}, F)| \leq d(n + n_0) \log p;$$

so $\widetilde{H}_{alg}(\overline{\alpha}, F) \leq d \log p$ for every $F \in \mathcal{F}(A)$, and we conclude that $\widetilde{\mathrm{ent}}(\alpha) \leq d \log p$. On the other hand, from (4.2) we get

$$\widetilde{\mathrm{ent}}(\overline{\alpha}) \geq \widetilde{H}_{alg}(\overline{\alpha}, q(\mathbb{F}_p)) = d \log p > 0.$$

If $\mathfrak{a}$ is not principal, pick a principal ideal $0 \neq \mathfrak{b} \leq \mathfrak{a}$. The above argument applied to $A' = R_p/\mathfrak{b}$ gives $\widetilde{\mathrm{ent}}(A') < \infty$. Since A is isomorphic to a quotient of A', by Fact 2.7 we conclude that $\widetilde{\mathrm{ent}}(A) \leq \widetilde{\mathrm{ent}}(A') < \infty$. On the other hand, by Lemma 5.4 there exists a principal ideal $\mathfrak{p}$ of R_p containing $\mathfrak{a}$. So $\widetilde{\mathrm{ent}}(R_p/\mathfrak{p}) > 0$ by the previous part of the proof, and hence $\widetilde{\mathrm{ent}}(R_p/\mathfrak{a}) > \widetilde{\mathrm{ent}}(R_p/\mathfrak{p}) > 0$ by Fact 2.7. □

Example 5.6. Consider the ideal $\mathfrak{a} = (X_1 - X_2)$ of $R_p = \mathbb{F}_p[X_1, X_2]$. Then the actions of X_1 and X_2 on the R_p-module $A = R_p/\mathfrak{a}$ are the same, say α. This means that, calling ϕ the multiplication by X_1 (or X_2) in A and of α_ϕ the relative $\mathbb{N}$-action on A, α coincides with the co-diagonal action action $\mathbb{N}^2 \overset{\alpha_\phi^{(2)}}{\curvearrowright} A$ of α_ϕ, defined by $\alpha_\phi^{(2)}(n,m) = \alpha_\phi(n+m) = \phi^{n+m}$ for every $n, m \in \mathbb{N}$. By Theorem 5.5, $\widetilde{\mathrm{ent}}(\alpha_\phi^{(2)}) = \widetilde{\mathrm{ent}}(A) = \log p$.

Now we describe when $\widetilde{\mathrm{ent}}(R_p/\mathfrak{a}) < \infty$ for a non-zero ideal $\mathfrak{a}$ of $R_p = \mathbb{F}_p[X_1, \ldots, X_m]$.

Corollary 5.7. *Let $R_p = \mathbb{F}_p[X_1, \ldots, X_m]$ for some $m > 1$. Then the following conditions are equivalent:*

(a) $m = 2$;

(b) there exists a principal ideal $\mathfrak{a} \neq 0$ of R_p such that $\widetilde{\mathrm{ent}}(R_p/\mathfrak{a}) < \infty$;

(c) $0 < \widetilde{\mathrm{ent}}(R_p/\mathfrak{a}) < \infty$ for every ideal $\mathfrak{a} \neq 0$ of R_p.

Proof. (a)⇒(c) follows from Theorem 5.5, (c)⇒(b) is trivial and (b)⇒(a) follows from Theorem 5.1. □

To conclude, we obtain a complete description when $0 < \widetilde{\mathrm{ent}}(A) < \infty$ for a finitely generated R_p-module A for $R_p = \mathbb{F}_p[X_1, X_2]$.

Corollary 5.8. *For an infinite finitely generated R_p-module A the following conditions are equivalent:*

(a) $\mathrm{ent}(A) = 0$;

(b) $0 < \widetilde{\mathrm{ent}}(A) < \infty$;

(c) $\mathrm{rank}_{R_p}(A) = 0$.

Proof. (a)⇔(c) was proved in Corollary 4.5, and (b)⇒(a) follows from Remark 2.4. To prove (c)⇒(b) write $A = C_1 + \ldots + C_n$, where C_i are cyclic submodules of A. By hypothesis, for $i \in \{1, \ldots, n\}$ we can write $C_i \cong R_p/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i \neq 0$ of R_p. By Theorem 5.5, $\widetilde{\mathrm{ent}}(C_i) < \infty$ for $i \in \{1, \ldots, n\}$. For $A' = C_1 \times \ldots \times C_n$ we have $\widetilde{\mathrm{ent}}(A') < \infty$ by Fact 2.7. As A is a quotient of A', we conclude that $\widetilde{\mathrm{ent}}(A) < \infty$ by Fact 2.7. At least one of the cyclic submodules C_i is infinite, so $\widetilde{\mathrm{ent}}(A) \geq \widetilde{\mathrm{ent}}(C_i) > 0$ by Fact 2.7. □

Remark 5.9. If $S = \mathbb{F}_p[X_1, \ldots, X_m]$ for some $m > 1$ and $A = S/\mathfrak{a}$ for an ideal $\mathfrak{a} \neq 0$ of S contained in the maximal ideal $\mathfrak{m} = (X_1, \ldots, X_m)$, we conjecture that $0 < \widetilde{\mathrm{ent}}(A) < \infty$ if and only if $\dim A = 1$, where dim denotes the Krull dimension of the quotient ring A.

If $\mathfrak{a}$ is principal, then $\dim S/\mathfrak{a} = m - 1$ by Krull's Principal Ideal Theorem, so this conjecture is consistent with Corollary 5.7. On the other hand, this conjecture covers also Example 5.2(a), where $\dim S/\mathfrak{a} = 1$.

We conclude with the following open problem.

Question 5.10. Let p be a prime and $m > 1$ and integer. Is $\widetilde{\mathrm{ent}}$ a length function in the category of $\mathbb{F}_p[X_1, \ldots, X_m]$-modules?

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